9/2/23 MATH 2060A Tutorial

Announcements:

- HW 2 is due tomorrow at 11am.
- HW 3 is due 17/10 at 11am.
- After today, WEI Yunsong will be teaching tutorials 4-6.

$f \in C^n(I)$

Section 6.4.

<u>Recall</u>: Taylor's theorem: let $n \in \mathbb{N}$, $I = [a,b]$, $f : I \to \mathbb{R}$ s.t. $f', f'', \ldots, f^{(n)}$ exist and are cts on I. and that $f^{(n+1)}$ exists on (a,b). If $x_0 \in I$, then for any $x \in I$, there exists a point $c \in I$ between x and x_0 such that

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x) \text{ Taylor Polynomial}}$$

$$+ \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{R_n(x) \text{ - Remainder term in derivative/Lagrange form.}}$$

Newton's Method: Let $I=[a,b]$, $f:I\to\mathbb{R}$ be twice differentiable on I.
Sps $f(a)f(b)<0$ and that there are constants m, M s.t. $|f'(x)|\geq m>0$, $|f''(x)|\leq M$ for $x\in I$, and let $K=\frac{M}{2m}$. Then there exists a $I^*\subseteq I$ containing a zero r of f s.t. for any $x_1\in I^*$, the sequence (x_n) defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \text{ for all } n\in\mathbb{N}$$

belongs in I^*, and $x_n\to r$. Moreover

$$|x_{n+1}-r|\leq K|x_n-r|^2 \text{ for all } n\in\mathbb{N}.$$

Q16: Let $I\subseteq\mathbb{R}$ be open, let $f:I\to\mathbb{R}$ be differentiable on I, and suppose $f''(a)$ exists at $a\in I$. Show that

$$f''(a)=\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}$$

Give an example where this limit exists, but the function does not have a second derivative at a.

Hint: Use L'Hopital's Rule on RHS.

Pf: We want to use LHR. Since $f''(a)$ exists, there is a small nbhd of a in which $f'(x)$ exists and is cts at a.

So $f(a+h)-2f(a)+f(a-h)$, h^2 are both differentiable for h small enough.

$\lim_{h\to 0} h^2 = 0$. $\lim_{h\to 0} f(a+h) - 2f(a) + f(a-h) = 2f(a) - 2f(a) = 0$.

$(h^2)' \neq 0$ for $h>0$. So we can apply LHR (differentiating wrt. h),

$$\text{RHS} = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h}.$$

$$= \lim_{h\to 0} \frac{f'(a+h) - f'(a) + f'(a) - f'(a-h)}{2h}$$

$= f''(a)$ by definition of $f''(a)$.

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad a = 0$$

$$f'(x) = 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right).$$

$$f''(0) = \lim_{x\to 0} \frac{3x^{\cancel{2}} \sin\left(\frac{1}{x}\right) - \cancel{x}\cos\left(\frac{1}{x}\right)}{\cancel{x}} = \lim_{x\to 0} 3x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \text{ DNE}.$$

But $$\lim_{h\to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} = \lim_{h\to 0} \frac{h^3 \sin\left(\frac{1}{h}\right) - (-h)^3 \sin\left(\frac{1}{-h}\right)}{h^2}$$

$$= \lim_{h\to 0} \frac{2h^{\cancel{3}} \sin\left(\frac{1}{h}\right)}{\cancel{h^2}} = \lim_{h\to 0} 2h\sin\left(\frac{1}{h}\right) = 0.$$

//.

Q18: Let $I \subseteq \mathbb{R}$, $c \in I$. Sps that f, g defined on I and that the derivatives $f^{(n)}, g^{(n)}$ exist and are cts on I. If $f^{(k)}(c) = g^{(k)}(c) = 0$ for $k = 0, 1, \ldots, n-1$ but $g^{(n)}(c) \neq 0$, show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

Pf: Apply is to Taylor's Thm to f, g at $x_0 = c$.

$$\frac{f(x)}{g(x)} = \frac{\displaystyle\sum_{k=0}^{n-1} \frac{\cancel{f^{(k)}(c)}^{\to 0}}{k!}(x-c)^k + \frac{f^{(n)}(z_1)}{\cancel{n!}}\cancel{(x-c)^n}}{\displaystyle\sum_{k=0}^{n-1} \frac{\cancel{g^{(k)}(c)}^{\to 0}}{k!}(x-c)^k + \frac{g^{(n)}(z_2)}{\cancel{n!}}\cancel{(x-c)^n}}$$

for some z_1 between x and c.

for some z_2 between x and c.

$$= \frac{f^{(n)}(z_1)}{g^{(n)}(z_2)}.$$

Because z_1, z_2 are between x and c,

by cty of $f^{(n)}$, $g^{(n)}$,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f^{(n)}(z_1)}{g^{(n)}(z_2)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

$\forall \varepsilon, \exists \delta$ s.t. $|x-c| < \delta_1$ ...

say if $|x-c| < \delta$, then $|z_1 - c| < \delta_1$, $|z_2 - c| < \delta_2$

Q19: Show that the function $f(x) = x^3 - 2x - 5$ has a zero r in $[2, 2.2]$.

If $x_1 = 2$ and we define the sequence (x_n) using Newton's method, show that $|x_{n+1} - r| \leq (0.7)|x_n - r|^2$

Compute x_4. (x_4 is accurate to within six decimal places).

Pf: f cts. $f(2) = -1 < 0$. $f(2.2) = 1.248 > 0$.

So f has a zero r in $[2, 2.2]$.

$|f'(x)| = |3x^2 - 2| \geq |3\cdot 2^2 - 2| = 10$.

$|f''(x)| = |6x| \leq |6 \cdot 2.2| \leq 13.2$.

So $K = \frac{M}{2m} = \frac{13.2}{20} = 0.66$.

So we have $|x_{n+1} - r| \leq 0.7 |x_n - r|^2$.

$x_1 = 2$.

$x_2 = 2 - \frac{f(2)}{f'(2)} = 2.1$

$x_3 = 2.1 - \frac{f(2.1)}{f'(2.1)} = \frac{11761}{5615} = 2.0945681211\ldots$

$x_4 = \frac{11761}{5615} - \frac{f\left(\frac{11761}{5615}\right)}{f'\left(\frac{11761}{5615}\right)} = 2.0945515\ldots$